

Husimi distribution, Wehrl entropy and superradiant phase in spin-boson interactions

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We study the Husimi distribution of the ground state in the Dicke model of field-matter interactions to visualize the quantum phase transition, from normal to superradiant, in phase-space. We follow an exact numerical and variational analysis, without making use of the usual Holstein-Primakoff approximation. We find that Wehrl entropy of the Husimi distribution provides an indicator of the sharp change of symmetry through the critical point. Additionally, we note that the zeros of the Husimi distribution characterize the Dicke model quantum phase transition.

I. INTRODUCTION

The study of quantum phase transitions (QPTs) is an important subject in many-body quantum physics [1]. If we consider a quantum system described by the Hamiltonian $H = H_0 + \lambda H_1$, where H_0 and H_1 have different symmetries and λ is a control parameter, QPT occurs when λ reaches a critical value λ_c for which the properties of the system change suddenly.

In this work we will analyze phase-space properties for a QPT and, for this purpose, we will consider the representative Dicke model of spin-boson interactions (see e.g. [2–5]). There are several distributions to analyze phase-space properties [6]; the most popular one is the Wigner distribution, but there is another important one, the Husimi distribution, which has the interesting property of non-negativity and it is defined as the overlap between a minimal uncertainty (coherent) state and the wavefunction. Recently, we have proposed the Husimi distribution as a tool for a phase-space visualization of QPTs using two algebraic models to exemplify the study: the Dicke model [7] and the vibron model [8], this last used to study rotational and vibrational spectra in diatomic and polyatomic molecules, which also exhibit a (shape) QPT. In Ref. [7] we made use of the Holstein-Primakoff approximation [9] (large spin j) to approximate the atomic sector by an harmonic oscillator for a large number of atoms $N = 2j$. Here we won't use this approximation and work with finite N in an exact manner.

The advantage of working in phase space is that we can analyze contributions in position and momentum space jointly. Additionally, we have characterized QPTs using the zeros of the Husimi distribution. Other information theoretic measures for QPT's in the Dicke and vibron models have been recently studied in position and/or momentum spaces, separately. In particular, it has been

shown that there is an abrupt change of the Rényi entropy [10], Fisher information [11] and complexity measures [12] at the transition point in the Dicke model. Moreover, it has been found that uncertainty Shannon [13] and Rényi [14, 15] entropic relations accounts for the QPTs better than other variance-based uncertainty relations. See also [16] for a recent paper on vibration-rotation entanglement measures of vibron models in the 'rigidly bent' phase.

The structure of the paper is the following. In Section II we briefly remind the Dicke model, boson and spin- j coherent states and we present the Husimi distribution (without the Holstein-Primakoff approximation) and the Wehrl entropy. In Section III we will present numerical and variational results in terms of symmetry-adapted coherent states. Three-dimensional plots, contour lines and Wehrl entropy of the Husimi distribution reveal a drastic change in the symmetry of the ground state wave function and provide a signature for the QPT even for a finite number of particles. Finally, zeros of the Husimi distribution (in the variational approximation) are also computed and graphically represented to characterize the QPT.

II. DICKE HAMILTONIAN, HUSIMI DISTRIBUTION AND WEHRL ENTROPY

The single-mode Dicke model is a well studied object in the field of QPTs [2, 3, 5]. In this case the Hamiltonian is given by

$$H = \omega_0 J_z + \omega a^\dagger a + \frac{\lambda}{\sqrt{2j}}(a^\dagger + a)(J_+ + J_-), \quad (1)$$

describing an ensemble of N two-level atoms with level-splitting ω_0 , with J_z , J_\pm the angular momentum operators for a pseudospin of length $j = N/2$, and a and a^\dagger

are the bosonic operators of the field with frequency ω . It is well known that there is a QPT at the critical value of the coupling parameter $\lambda = \lambda_c = \frac{\sqrt{\omega\omega_0}}{2}$ from the so-called normal phase ($\lambda < \lambda_c$) to the superradiant phase ($\lambda > \lambda_c$).

Let us consider a basis set $\{|n; j, m\rangle \equiv |n\rangle \otimes |j, m\rangle\}$ of the Hilbert space, with $\{|n\rangle\}_{n=0}^\infty$ the number states of the field and $\{|j, m\rangle\}_{m=-j}^j$ the so called Dicke states of the atomic sector. The matrix elements of the Hamiltonian in this basis are:

$$\begin{aligned} \langle n'; j', m' | H | n; j, m \rangle &= (n\omega + m\omega_0) \delta_{n',n} \delta_{m',m} \\ &+ \frac{\lambda}{\sqrt{2j}} (\sqrt{n+1} \delta_{n',n+1} + \sqrt{n} \delta_{n',n-1}) \\ &\times (\sqrt{j(j+1) - m(m+1)} \delta_{m',m+1} \\ &+ \sqrt{j(j+1) - m(m-1)} \delta_{m',m-1}). \end{aligned} \quad (2)$$

At this point it is important to note that time evolution preserves the parity $e^{i\pi(n+m+j)}$ of a given state $|n; j, m\rangle$. That is, the parity operator $\hat{\Pi} = e^{i\pi(a^\dagger a + J_z + j)}$ commutes with H and both operators can then be jointly diagonalized. In particular, the ground state must be even (see later on Eq. (13)).

Let us denote by

$$\begin{aligned} |\alpha\rangle &= e^{-|\alpha|^2/2} e^{\alpha a^\dagger} |0\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^\infty \frac{\alpha^n}{\sqrt{n!}} |n\rangle, \\ |z\rangle &= (1 + |z|^2)^{-j} e^{z J_+} |j, -j\rangle = \\ &(1 + |z|^2)^{-j} \sum_{m=-j}^j \binom{2j}{j+m}^{1/2} z^{j+m} |j, m\rangle, \end{aligned} \quad (3)$$

(with $\alpha, z \in \mathbb{C}$) the standard (canonical or Glauber) and spin- j Coherent States (CSs) for the photon and the particle sectors, respectively. It is well known (see e.g. [17]) that coherent states form an overcomplete set of the corresponding Hilbert space and fulfill the closure relations or resolutions of the identity:

$$\begin{aligned} 1 &= \frac{1}{\pi} \int_{\mathbb{R}^2} |\alpha\rangle \langle \alpha| d^2\alpha, \\ 1 &= \frac{2j+1}{\pi} \int_{\mathbb{R}^2} |z\rangle \langle z| \frac{d^2z}{(1 + |z|^2)^2}, \end{aligned} \quad (4)$$

with $d^2w \equiv d\text{Re}(w) d\text{Im}(w)$ (or $d^2w = r dr d\theta$ in polar coordinates $w = re^{i\theta}$) the Lebesgue measure on \mathbb{C} . The complex parameters α and z are related to the mean number of photons, as $\langle \alpha | a^\dagger a | \alpha \rangle = |\alpha|^2$, and the mean fraction of excited atoms, as $\langle z | J_z + j | z \rangle = N |z|^2 / (1 + |z|^2)$, respectively. It is also straightforward to see that the probability amplitude of detecting n photons and $j+m$ excited atoms in $|\alpha, z\rangle \equiv |\alpha\rangle \otimes |z\rangle$ is given by:

$$\varphi_{n,m}^{(j)}(\alpha, z) = \langle n | \alpha \rangle \langle j, m | z \rangle = \frac{e^{-|\alpha|^2/2} \alpha^n \sqrt{\binom{2j}{j+m}} z^{j+m}}{\sqrt{n!} (1 + |z|^2)^j}. \quad (5)$$

The ground state vector ψ will be given as an expansion

$$|\psi\rangle = \sum_{n=0}^{n_c} \sum_{m=-j}^j c_{nm}^{(j)} |n; j, m\rangle \quad (6)$$

where the coefficients $c_{nm}^{(j)}$ are calculated by numerical diagonalization of (2), with a given cutoff n_c , and depend on the control parameter λ . The Husimi distribution of ψ is then given by

$$\begin{aligned} \Psi(\alpha, z) &= |\langle \alpha, z | \psi \rangle|^2 \\ &= \sum_{n,n'=0}^{n_c} \sum_{m,m'=-j}^j c_{nm}^{(j)} \bar{c}_{n'm'}^{(j)} \varphi_{n,m}^{(j)}(\alpha, z) \varphi_{n',m'}^{(j)}(\bar{\alpha}, \bar{z}) \end{aligned} \quad (7)$$

and normalized according to:

$$\int_{\mathbb{R}^4} \Psi(\alpha, z) d\mu(\alpha, z) = 1, \quad (8)$$

with integration measure:

$$d\mu(\alpha, z) = \frac{2j+1}{\pi^2} \frac{d^2\alpha d^2z}{(1 + |z|^2)^2}. \quad (9)$$

An important quantity to visualize the QPT in the Dicke model across the critical point λ_c will be the Wehrl entropy

$$W_j(\lambda) = - \int_{\mathbb{R}^4} \Psi(\alpha, z) \ln(\Psi(\alpha, z)) d\mu(\alpha, z), \quad (10)$$

where the dependence of W_j on λ comes from the dependence of $c_{nm}^{(j)}$ on λ .

III. NUMERICAL VERSUS VARIATIONAL RESULTS

In Figure 1 we represent a 3D plot of the exact Husimi distribution of the ground state $\Psi(\alpha, z)$ in ‘position’ (α and z real) and ‘momentum’ (α and z imaginary) spaces. We observe that the Husimi distribution in position space is concentrated around $\alpha = 0 = z$ at the normal phase $\lambda < \lambda_c$ (no photons and no excited atoms) but splits into two differentiated packets at the superradiant phase $\lambda > \lambda_c$. In momentum space, the Husimi distribution becomes more and more delocalized with the emergence of multiple modulations above the critical point λ_c (see also later on Figure 3 for a contour line of the variational case).

This delocalization of the exact Husimi distribution is captured by the Wehrl entropy $W_j(\lambda)$ as a function of λ for different values of j . The computed results are given in Fig. 2, where we present $W_j(\lambda)$ for $j = 5$ and $j = 10$ (solid lines) and for $\omega = \omega_0 = 1$ (for which $\lambda_c = 0.5$), together with the variational results (see later). The Wehrl entropy tends to 2 (for high j) in the normal phase, and to $2 + \ln 2$ in the superradiant phase, with an abrupt change (more abrupt as j increases) around the critical point.

The exact values of $W_j(\lambda)$ for $\lambda \ll \lambda_c$ and $\lambda \gg \lambda_c$ are nicely reproduced by the following trial states expressed in terms of ‘parity-symmetry-adapted’ CSs introduced

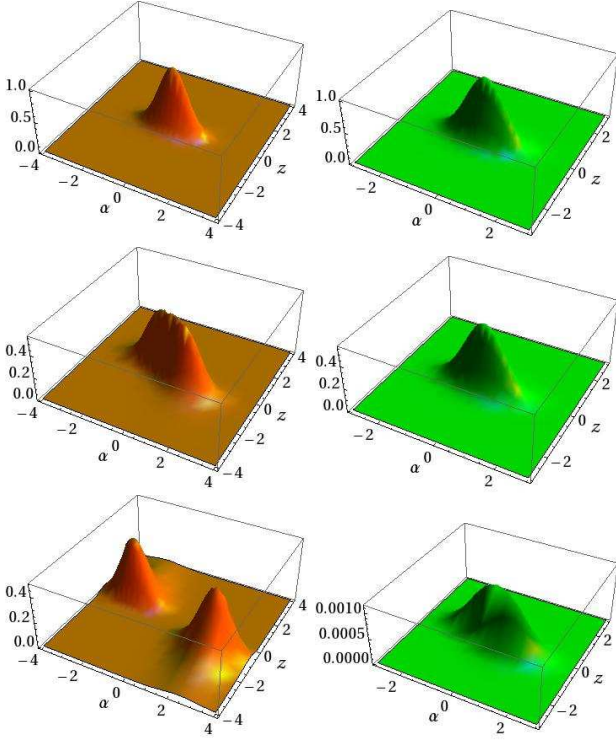


FIG. 1: (Color online) 3D-Plot of the exact Husimi distribution in (left) “position space” (α and z real), and (right) “momentum space” (α and z imaginary) for different values of λ (from top to bottom: $\lambda = 0$, $\lambda = 0.6$ and $\lambda = 1$) for $j = 3$ and $\omega = \omega_0 = 1 \Rightarrow \lambda_c = 0.5$. Atomic units.

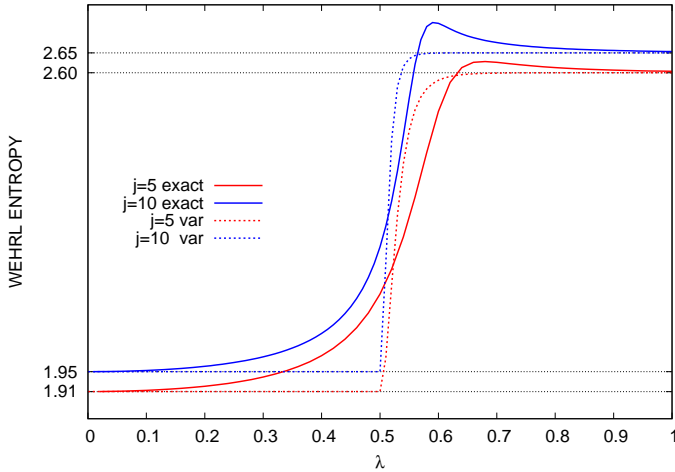


FIG. 2: (Color online) Exact (solid) and variational (dotted) Wehrl entropies $W_j(\lambda)$ for $j = 5$ and $j = 10$ ($\omega_0 = \omega = 1 \Rightarrow \lambda_c = 0.5$) as a function of λ . Wehrl entropy $W_j(\lambda)$ grows with the number of atoms $N = 2j$ and the control parameter λ , attaining the limit values $W_j(0) = 1 + 2j/(2j+1) \xrightarrow{j \rightarrow \infty} 2$ and $W_\infty(\infty) = 2 + \ln(2)$ in the thermodynamic limit.

by Castaños et al. [18, 19], which turn out to be an excellent approximation to the exact quantum solution of the ground (+) and first excited (−) states of the Dicke model.

Using the direct product $|\alpha, z\rangle \equiv |\alpha\rangle \otimes |z\rangle$ as a ground-state ansatz, one can easily compute the mean energy

$$\mathcal{H}(\alpha, z) = \langle \alpha, z | H | \alpha, z \rangle = \omega |\alpha|^2 + j\omega_0 \frac{|z|^2 - 1}{|z|^2 + 1} + \lambda \sqrt{2j} (\alpha + \bar{\alpha}) \frac{\bar{z} + z}{|z|^2 + 1}, \quad (11)$$

which defines a four-dimensional “energy surface”. Minimizing with respect to these four coordinates gives the equilibrium points (see [18, 19]):

$$\alpha_e = \begin{cases} 0, & \text{if } \lambda < \lambda_c \\ -\sqrt{2j} \sqrt{\frac{\omega_0}{\omega}} \frac{\lambda}{\lambda_c} \sqrt{1 - \left(\frac{\lambda}{\lambda_c}\right)^{-4}}, & \text{if } \lambda \geq \lambda_c \end{cases} \quad (12)$$

$$z_e = \begin{cases} 0, & \text{if } \lambda < \lambda_c \\ \sqrt{\frac{\frac{\lambda}{\lambda_c} - \left(\frac{\lambda}{\lambda_c}\right)^{-1}}{\frac{\lambda}{\lambda_c} + \left(\frac{\lambda}{\lambda_c}\right)^{-1}}}, & \text{if } \lambda \geq \lambda_c \end{cases}$$

Note that α_e and z_e are real and non-zero above the critical point λ_c (i.e., in the superradiant phase).

Although the direct product $|\alpha, z\rangle$ gives a good variational approximation to the ground state mean energy in the thermodynamic limit $j \rightarrow \infty$, it does not capture the correct behavior for other ground state properties sensitive to the parity symmetry $\hat{\Pi}$ of the Hamiltonian (1) like, for instance, uncertainty and entropy measures [13, 14]. This is why parity-symmetry-adapted coherent states are introduced. Indeed, a far better variational description of the ground (resp. first-excited) state is given in terms of the even-(resp. odd)-parity coherent states [18, 19]

$$|\psi_\pm\rangle = |\alpha, z, \pm\rangle = \frac{|\alpha\rangle \otimes |z\rangle \pm |-\alpha\rangle \otimes | -z\rangle}{\mathcal{N}_\pm(\alpha, z)}, \quad (13)$$

obtained by applying projectors of even and odd parity $\hat{\mathcal{P}}_\pm = (1 \pm \hat{\Pi})$ to the direct product $|\alpha\rangle \otimes |z\rangle$. Here

$$\mathcal{N}_\pm(\alpha, z) = \sqrt{2} \left(1 \pm e^{-2|\alpha|^2} \left(\frac{1 - |z|^2}{1 + |z|^2} \right)^{2j} \right)^{1/2} \quad (14)$$

is a normalization factor. These even and odd coherent states are “Schrödinger cat states” in the sense that they are a quantum superposition of quasi-classical, macroscopically distinguishable states. The new energy surface $\mathcal{H}_\pm(\alpha, z) = \langle \alpha, z, \pm | H | \alpha, z, \pm \rangle$ (see [18, 19] for an explicit expression of it) is more involved than $\mathcal{H}(\alpha, z)$ in (11) and makes much more difficult to obtain the new critical points $\alpha_e^{(\pm)}, z_e^{(\pm)}$ minimizing the corresponding energy surface. The reader is addressed to Ref. [20] in this volume for a numerical computation of the new critical points. It should be emphasized that the equilibrium points given in the expression (12) are correct only in the thermodynamic limit $j \rightarrow \infty$ or far from $\lambda = \lambda_c$ for finite j . Otherwise the minimization of $\mathcal{H}_\pm(\alpha, z)$ should be

done (see [18–20] for more details). In this paper, instead of carrying out a numerical computation of $\alpha_e^{(\pm)}, z_e^{(\pm)}$ for different values of j and λ , we shall use the approximation $\alpha_e^{(\pm)} \approx \alpha_e, z_e^{(\pm)} \approx z_e$, which turns out to be quite good except in a close neighborhood around λ_c , which diminishes as the number of particles $N = 2j$ increases (see Refs. [19, 20]). With this approximation, we expect a rather good agreement between our numerical and variational results except perhaps in a close vicinity of λ_c (indeed, see Figure 2).

Taking into account the coherent state overlaps

$$\begin{aligned} \langle \alpha | \pm \alpha_e \rangle &= e^{-\frac{1}{2}|\alpha| - \frac{1}{2}\alpha_e^2 \pm \bar{\alpha}\alpha_e}, \\ \langle z | \pm z_e \rangle &= \frac{(1 \pm \bar{z}z_e)^{2j}}{(1 + |z|^2)^j(1 + z_e^2)^j}, \end{aligned} \quad (15)$$

the Husimi distribution for the variational states $|\alpha_e, z_e, \pm\rangle$ can be simply written as:

$$\Psi_{\pm}(\alpha, z) = \frac{|\langle \alpha, z | \alpha_e, z_e \rangle \pm \langle \alpha, z | -\alpha_e, -z_e \rangle|^2}{\mathcal{N}_{\pm}^2(\alpha_e, z_e)}. \quad (16)$$

From now on we shall restrict ourselves to the even case and simply denote by $\Psi = \Psi_+$ the Husimi distribution of the variational ground state. Figure 3 shows a contour line of the variational Husimi distribution. Note that, in position space, it reproduces the packet splitting across the critical point depicted in Figure 1, with two differentiated packets located around the equilibrium points (α_e, z_e) and its antipode $(-\alpha_e, -z_e)$ in the superradiant phase. In momentum space, it exhibits a delocalization and ‘modulation’ for increasing values of λ . We can easily compute the Wehrl entropy of (16), which gives

$$W_j(\lambda) = \begin{cases} 1 + \frac{2j}{2j+1}, & \text{if } \lambda < \lambda_c \\ 1 + \frac{2j}{2j+1} + \ln 2, & \text{if } \lambda \gg \lambda_c. \end{cases} \quad (17)$$

denoting an entropy excess of $\ln(2)$ in the superradiant phase. In the normal phase we have exactly $W_j(\lambda) = 1 + 2j/(2j+1)$, as corresponds to a coherent state according to the (still unproved) Lieb’s conjecture. Indeed, as conjectured by Wehrl [21] and proved by Lieb [22], any Glauber coherent state $|\alpha\rangle$ has a minimum Wehrl entropy of 1. In the same paper by Lieb [22], it was also conjectured that the extension of Wehrl’s definition of entropy for coherent spin- j states $|z\rangle$ will yield a minimum entropy of $2j/(2j+1)$. For the joined system of radiation field plus atoms we would have $W_j(\lambda) = 1 + 2j/(2j+1)$ in the normal phase ($\lambda < \lambda_c$), and therefore, $W_j \rightarrow 2$ in the thermodynamic limit $j \rightarrow \infty$, in agreement with our result.

To finish, we would like to comment on the zeros of the Husimi distribution as a fingerprint for the QPT (see [7] for more information). From (16) we obtain

$$\Psi(\alpha, z) = 0 \Rightarrow 2\alpha\alpha_e + 2j \ln \frac{1 + zz_e}{1 - zz_e} = i\pi(2l + 1), \quad l \in \mathbb{Z}, \quad (18)$$

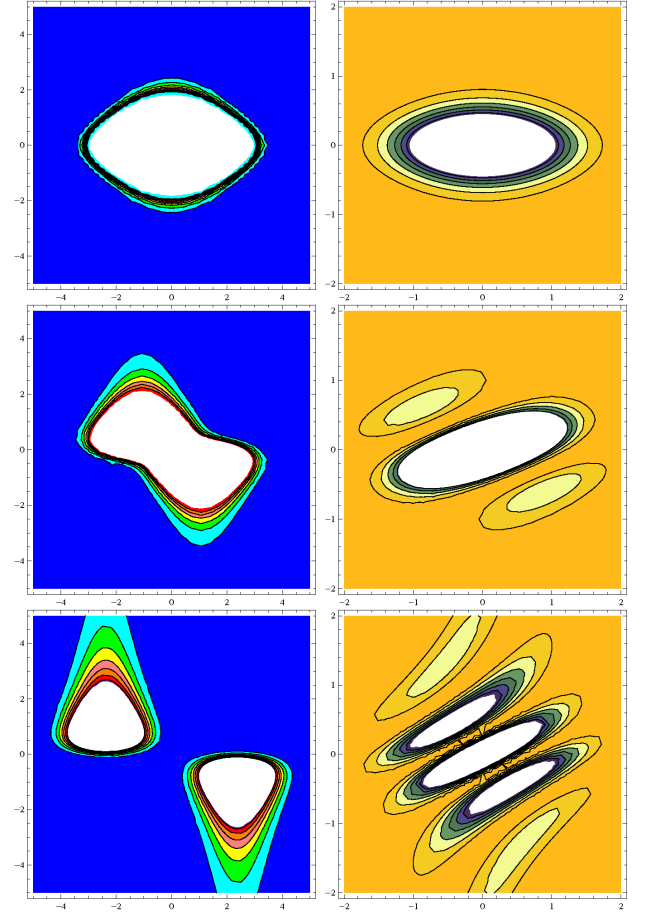


FIG. 3: (Color online) Contour lines of the variational Husimi distribution $\Psi_+(\alpha, z)$ in “position space” (α and z real; left panel) and “momentum space” (α and z imaginary; right panel) for different values of λ (from top to bottom: $\lambda = 0$, $\lambda = 0.6$ and $\lambda = 1$) for $j = 3$ and $\omega = \omega_0 = 1 \Rightarrow \lambda_c = 0.5$. Atomic units.

which defines a two-dimensional surface (for each value of l) in a four-dimensional manifold with parametric equations:

$$\alpha = f_j^{(l)}(z, \lambda) = \frac{j}{\alpha_e} \ln \frac{1 - zz_e}{1 + zz_e} + \frac{i\pi}{2\alpha_e}(2l + 1). \quad (19)$$

This expression gives in particular the “less probable mean photon number $|\alpha|^2$ for each mean atom fraction $|z|^2/(1 + |z|^2)$ ” in phase space (remember comment before Eq. (5)). In Figure 4 we represent this surface as a conformal mapping of a regular grid in the z -plane. That is, for $z = z_1 + iz_2$, we represent the image of vertical lines $z_1 = \text{constant}$ (solid-red curves) and horizontal lines $z_2 = \text{constant}$ (dotted-blue curves). We see from (18) that, in the normal phase ($\alpha_e = 0 = z_e$) the Husimi distribution $\Psi(\alpha, z)$ has no zeros. In the superradiant phase ($\lambda > \lambda_c$) there are more and more zeros as j and λ increase. To study the high j limit, we can redefine

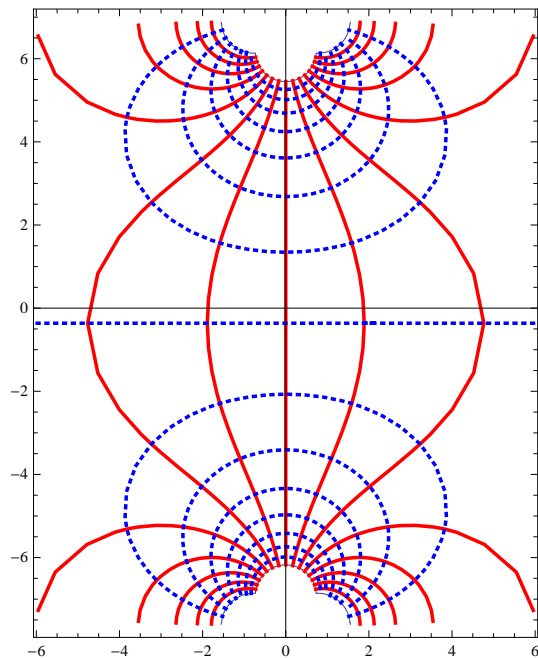


FIG. 4: (Color online) Surface of zeros $\alpha = f_j^{(l)}(z, \lambda)$ of the variational Husimi distribution $\Psi_+(\alpha, z)$ for $\lambda = 1, j = 10$ and $l = 0$ ($\lambda_c = 0.5$) seen as a conformal mapping of a regular grid in the z -plane.

$\beta \equiv \sqrt{2j} z$, which simplifies the expression of:

$$2j \ln \frac{1 + zz_e}{1 - zz_e} \simeq 2\beta\beta_e, \text{ for } j \gg 1, \quad (20)$$

where we have made use of the definition of the Euler number at some stage. Therefore, the equation (19) becomes:

$$\alpha_1 = -\frac{\beta_e}{\alpha_e} \beta_1, \quad \alpha_2 = -\frac{\beta_e}{\alpha_e} \beta_2 + \frac{\pi}{2\alpha_e} (2l + 1). \quad (21)$$

for $\alpha = \alpha_1 + i\alpha_2$ and $\beta = \beta_1 + i\beta_2$. Therefore, in the high j limit, and in the superradiant phase ($\lambda > \lambda_c$), the zeros are localized along straight lines (“dark fringes”) in the $\alpha_1\beta_1$ (position) and $\alpha_2\beta_2$ (momentum) planes. In the momentum plane, the number of dark fringes grows with λ and j . In the thermodynamic limit $j \rightarrow \infty$, zeros densely fill the momentum plane $\alpha_2\beta_2$ (see [7] for a graphical representation of zeros in the high j limit).

IV. CONCLUSIONS

We have found that Wehrl entropy of the Husimi distribution provides a sharp indicator of a quantum phase

transition in the Dicke model even for finite j . This uncertainty measure detects a delocalization of the Husimi distribution across the critical point λ_c and we have employed it, together with three-dimensional plots and contour lines of the Husimi distribution, to quantify and visualize the phase-space spreading of the ground state.

Calculations have been done numerically and through a variational approximation. The variational approach, in terms of symmetry-adapted coherent states, complements and enriches the analysis providing explicit analytical expressions for the Husimi distribution and Wehrl entropies which remarkably coincide with the numerical results, especially in the thermodynamic limit and far from $\lambda = \lambda_c$, where the approximate equilibrium points (12) fail. A more accurate calculation could be perhaps done by using the ‘true’ equilibrium points of Ref. [20], although we think our variational approach still captures the qualitative behavior near λ_c and the quantitative exact values far from λ_c (see again Figure 2 in this respect).

In the superradiant phase, Wehrl entropy undergoes an entropy excess of $\ln(2)$. This fact implies that the Husimi distribution splits up into two identical subpackets with negligible overlap in passing from normal to superradiant phase. In general, for s identical subpackets with negligible overlap, one would expect an entropy excess of $\ln(s)$.

The QPT fingerprints in the Dicke model have also been tracked by exploring the distribution of zeros of the Husimi density within the analytical variational approximation. Now, we have corroborated that the zeros of the Husimi distribution evidence the QPT without the Holstein-Primakoff approximation, founding again that there are no zeros in the normal phase and a larger number of zeros as j and λ increase in the superradiant phase. This interesting result supports the asseveration that the emergence of zeros of the Husimi distribution can be an indicator of QPTs [7, 8].

Acknowledgments

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